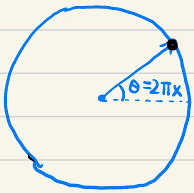
Fourier Analysis　　　Mar 19, 2024

Review:

Consider the heat equation on the circle

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, & x \in [0,1),\quad t > 0 \qquad (*) \\ u(x,0) = f(x). & \qquad (**) \end{cases}$$



$u = u(x,t)$ denotes the temperature at the point x and time t.

In the last lecture, we roughly derived the following formula of u by using the superposition method:

$$u(x,t) = \sum_{n=-\infty}^{\infty} \hat{f}(n)\, e^{-4\pi^2 n^2 t}\, e^{2\pi i n x}. \qquad (***)$$

Proposition 1: Let f be 1-periodic function on $\mathbb{R}$.

Assume that f is Riemann integrable on $[0,1)$.

Then

$$u(x,t) = \sum_{n \in \mathbb{Z}} \hat{f}(n)\, e^{-4\pi^2 n^2 t}\, e^{2\pi i n x} \quad \text{on} \quad \mathbb{R} \times (0,\infty)$$

satisfies $(*)$. Furthermore if f is cts at x_0, then $\lim_{t \to 0} u(x_0,t) = f(x_0)$.

Recall a useful result in Math 2060:

Thm. Let $J \subset \mathbb{R}$ be an interval. Let (f_n) be a sequence of diff functions on J.

Suppose ① $\exists\, x_0 \in J$ such that $f_n(x_0)$ converges as $n \to \infty$

② $f_n'(x) \rightrightarrows g(x)$ on J, as $n \to \infty$.

Then • $f_n(x) \rightrightarrows f(x)$ on J for some f as $n \to \infty$.

• $f'(x) = g(x)$ on J.

Pf of Prop 1.

Let $t_0 > 0$. Notice that

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)\, e^{-4\pi^2 n^2 t}\, e^{2\pi i n x}$$

converges uniformly on $\mathbb{R} \times (t_0, \infty)$ (by Weierstrass' M-test)

Hence u is cts on $\mathbb{R} \times (t_0, \infty)$.

Using the same argument, we see that

$$\sum_{n=-\infty}^{\infty} \frac{\partial}{\partial t}\left(\hat{f}(n)\, e^{-4\pi^2 n^2 t}\, e^{2\pi i n x} \right)$$

converges uniformly on $\mathbb{R} \times (t_0, \infty)$

Thus
$$\frac{\partial u(x,t)}{\partial t} = \sum_{n=-\infty}^{\infty} \frac{\partial}{\partial t}\left(\hat{f}(n) e^{-4\pi^2 n^2 t} e^{2\pi i n x}\right).$$

Similarly
$$\frac{\partial^2 u}{\partial x^2} = \sum_{n=-\infty}^{\infty} \frac{\partial^2}{\partial x^2}\left(\hat{f}(n) e^{-4\pi^2 n^2 t} e^{2\pi i n x}\right)$$

Since
$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\hat{f}(n) e^{-4\pi^2 n^2 t} e^{2\pi i n x}\right) \\
&= \frac{\partial^2}{\partial x^2}\left(\hat{f}(n) e^{-4\pi^2 n^2 t} e^{2\pi i n x}\right) \\
&= \hat{f}(n) \cdot (-4\pi^2 n^2)\, e^{-4\pi^2 n^2 t} e^{2\pi i n x},
\end{aligned}$$

we obtain
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \text{on} \quad \mathbb{R} \times (t_0, \infty)$$

Since t_0 is arbitrarily taken, so u satisfies (*) on $\mathbb{R} \times (0, \infty)$.

To see the limiting property of $u(x,t)$ as $t \to 0$, let us write

$$H_t(x) := \sum_{n=-\infty}^{\infty} e^{-4\pi^2 n^2 t} e^{2\pi i n x} \quad \text{on } \mathbb{R} \times (0, \infty)$$

We call it the heat kernel on the circle.

$(H_t)_{t>0}$ is a good kernel as $t \to 0$ in the following sense:

- $\int_0^1 H_t(x)\,dx = 1$ for all $t>0$. (easily checked)
- $H_t(x) > 0$, for all $t>0$ (will be checked in our later classes)
- $\forall\, \delta > 0$, $\int_\delta^{1-\delta} H_t(x)\,dx \to 0$ as $t \to 0$.

We claim that

$$\begin{aligned} u(x,t) &= H_t * f(x) \\ &= \int_0^1 f(x-y)\, H_t(y)\,dy. \end{aligned}$$

Notice that

$$\begin{aligned} \widehat{H_t * f}(n) &= \widehat{H_t}(n) \cdot \hat{f}(n) \\ &= e^{-4\pi^2 n^2 t} \cdot \hat{f}(n) \end{aligned}$$

$$\hat{u}_t(n) = e^{-4\pi^2 n^2 t}\hat{f}(n)$$

Since both $u(\cdot, t)$ and $H_t * f(\cdot)$ are cts in x and they have the same Fourier series, so

$$u(x,t) = H_t * f(x).$$

Since $(H_t)_{t>0}$ is a good kernel as $t \to 0$, so we get

$$\lim_{t \to 0} u(x,t) = f(x) \quad \text{provided that } f \text{ is cts at } x.$$

$\square$

Chap 5. The Fourier transform on $\mathbb{R}$.

A reasonable 2π-periodic function on $\mathbb{R}$ can be represented by its Fourier series.

Similarly a reasonable ℓ-periodic function on $\mathbb{R}$ can be represented by its Fourier series.

For instance, if f is an ℓ-periodic diff function on $\mathbb{R}$, then

$$f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n)\, e^{\frac{2\pi}{\ell} i n x}$$

where $\hat{f}(n) = \frac{1}{\ell} \int_0^{\ell} f(x)\, e^{-\frac{2\pi}{\ell} i n x}\, dx$.

- Do we have an analogue for non-periodic functions on $\mathbb{R}$?

§ 5.1 Functions of moderate decrease and integrations.

Def. A function $f : \mathbb{R} \to \mathbb{C}$ is said to be of moderate decrease if

① f is cts on $\mathbb{R}$.

② $\exists\ A > 0$ such that

$$|f(x)| \leq \frac{A}{1+x^2} \quad \text{for all } x \in \mathbb{R}.$$

For convenience, we use $\mathcal{M}(\mathbb{R})$ denote the collection of all functions of moderate decrease.

Then $\mathcal{M}(\mathbb{R})$ is a vector space.

If $f, g \in \mathcal{M}(\mathbb{R})$, then $\alpha f + \beta g \in \mathcal{M}(\mathbb{R})$ for all $\alpha, \beta \in \mathbb{C}$.

Def. (improper integration)

Let $f \in M(\mathbb{R})$. We define

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{N\to\infty} \int_{-N}^{N} f(x)\,dx .$$

Lemma 1. Let $f \in M(\mathbb{R})$. Then the above limit exists.

Pf. Write $I_N := \int_{-N}^{N} f(x)\,dx$, $N \in \mathbb{N}$.

To show the limit exists, we only need to show that (I_N) is a Cauchy sequence.

Let $M > N$.

$$|I_M - I_N| = \left| \int_{|x|\le M} f(x)\,dx - \int_{|x|\le N} f(x)\,dx \right|$$

$$= \left| \int_{N<|x|\le M} f(x)\,dx \right|$$

$$\leq \int_{N<|x|\leq M} |f(x)|\,dx$$

$$\leq \int_{|x|>N} \frac{A}{1+x^2}\,dx$$

$$\leq A \int_{|x|>N} \frac{1}{x^2}\,dx$$

$$= \frac{2A}{N} \longrightarrow 0 \quad \text{as } N \to \infty$$

So (I_N) is a Cauchy sequence. $\square$

Lemma 2. We write

$$L(f) = \int_{-\infty}^{\infty} f(x)\,dx \quad \text{for } f \in \mathcal{M}(\mathbb{R}).$$

Then

① L is linear, i.e.

$$L(\alpha f + \beta g) = \alpha L(f) + \beta L(g)$$

for $f, g \in \mathcal{M}(\mathbb{R})$ and $\alpha, \beta \in \mathbb{C}$.

② L is translation invariant.

$$\int_{-\infty}^{\infty} f(x+h)\,dx = \int_{-\infty}^{\infty} f(x)\,dx, \quad \forall\, h \in \mathbb{R}.$$

③ Scaling under dilation: $\forall\, \delta > 0$,

$$\delta \int_{-\infty}^{\infty} f(\delta x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx.$$

④ Absolute continuity.

$$\lim_{h \to 0} \int_{-\infty}^{\infty} |f(x+h) - f(x)|\,dx = 0$$

Pf of ④. Let $\varepsilon > 0$. Assume that $h \in (-1, 1)$.

Take a large number $N > 0$ such that

$$\int_{|x| \geq N-1} |f(x)|\, dx < \varepsilon.$$

Then for $h \in (-1, 1)$, we have

$$\int_{|x| \geq N} |f(x+h)|\, dx$$

$$\leq \int_{|x| \geq N-1} |f(x)|\, dx < \varepsilon.$$

Hence

$$\int_{-\infty}^{\infty} |f(x+h) - f(x)|\, dx = \int_{|x| \geq N} |f(x+h) - f(x)|\, dx + \int_{|x| \leq N} |f(x+h) - f(x)|\, dx$$

$$\leq \int_{|x| \geq N} |f(x+h)|\, dx + \int_{|x| \geq N} |f(x)|\, dx + \int_{|x| \leq N} |f(x+h) - f(x)|\, dx$$

$$\leq \quad 2\varepsilon + \int_{|x| \leq N} |f(x+h) - f(x)|\, dx$$

for all $h \in (-1, 1)$.

Notice that f is uniformly cts on $[-N-1, N+1]$.

Hence $\exists\ \delta > 0$ such that

$$|f(x+h) - f(x)| \leq \frac{\varepsilon}{2N}$$

for all $x \in [-N, N]$ and $|h| < \delta$.

So for h with $|h| < \delta$,

$$\int_{|x| \leq N} |f(x+h) - f(x)|\, dx \leq \frac{\varepsilon}{2N} \cdot 2N = \varepsilon.$$

Therefore

$$\int_{-\infty}^{\infty} |f(x+h) - f(x)|\, dx < 3\varepsilon \quad \text{if } |h| < \delta.$$

$\boxtimes$